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Ginzburg–Landau system of complex modulation equations for a distributed nonlinear-dispersive transmission line

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Abstract

This work is devoted to the investigation of a nonlinear transmission line containing nonlinear capacitors. In this work, we study the stability of a set of two coupled Ginzburg–Landau (GL) equations derived from a model of a nonlinear transmission line. After deriving the main differential equation for the voltage, we consider an expansion of the voltage amplitudes for two travelling waves and obtain the time and space Ginzburg–Landau differential equations for these amplitudes. We next study the existence and stability of the modulated amplitude waves in the complex plane, and show the existence of solition-like solutions.

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I dedicate this work to my children Kengneson Cris-Carelle Djiké, Kengneson Weierstrass Owan Wambo, and Kengneson Delma Djomo

1. Introduction: description of the model

1.1. Introduction

The theory of transmission lines is a classical topic of electrical engineering. Recently, this topic has received renewed attention and has been a focus of considerable research. This is because the transmission line theory has found new and important applications in the area of high-speed VLSI interconnects, and it has also retained its significance in the area of power transmission. In many applications, transmission lines are connected to nonlinear circuits. The study of nonlinear wave propagation along distributed electrical lines is important because

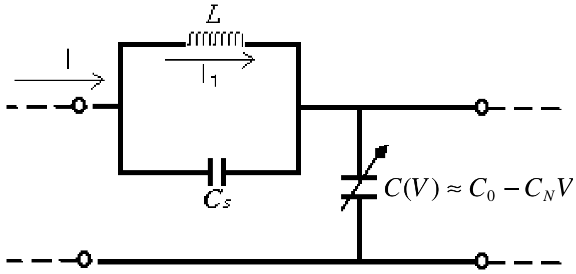

Figure 1. A section of a distributed nonlinear dispersive-transmission line.

these lines serve as useful models for nonlinear dispersive wave motion in many interesting physical systems [1–4]. In a recent paper, Kengne [5] showed that wave modulation in the discrete nonlinear RLC transmission line is governed by a cubic complex Ginzburg–Landau equation [6], in which the complex amplitude appears.

The Ginzburg–Landau (GL) equation [7] is the appropriate amplitude equation to describe the slow dynamics near a super critical transition to unidirectional travelling waves. It is a generic nonlinear model with various physical applications, including binary-fluid thermal convection [8], semiconductor lasers [9], etc. Ginzburg–Landau equations also represent a class of universal mathematical models which describe pattern formation in various nonlinear media. One of the most fundamental types of patterns are solitary pulses (SPs), often called solitons in loose terms. In particular, the simplest generic species of the GL equations, namely the cubic complex Ginzburg–Landau equation (CGLE), gives rise to a well-known exact SP solution [10]. However, this solution is unstable (as the equation includes a linear gain term, which makes the zero solution unstable, precluding stability of any solitary pattern). Therefore, the search for physically relevant models of the GL type that give rise to stable pulses has attracted much attention. One possibility is to introduce a cubic–quintic GL equation [11] with linear loss and cubic gain, nonlinear stability being provided by a quintic loss term.

In this paper, we follow Kengne [11] and consider a distributed nonlinear dispersive-transmission LC line. We focus on the case of two wavepackets. Assuming the spatial and temporal modulation of the solutions, the interaction of the modes can be described by two cubic coupled Ginzburg–Landau equations. Had we considered one wavepacket, we would have found that the spatial and temporal evolution of its amplitude and phase is described by a single complex Ginzburg–Landau equation.

Searching for coherent structures allows one to reduce a partial differential equation to an ordinary one, and such solutions of the CGLE are believed to be extremely important in many regimes, including spatiotemporal chaos [12].

Although many studies have been undertaken in various systems showing that the dynamics of nonlinear excitations are governed by the complex Ginzburg–Landau equations or systems, to our knowledge, no work using the mono-inductance nonlinear-dispersive transmission line has been reported which shows that solitons can exist.

1.2. Description of the model

We consider a discrete nonlinear mono-inductance transmission line shown in figure 1. In this transmission line C_N is a nonlinear capacitor such as a 'VARICAP' or a reverse-biased p–n junction diode, the capacitance of which depends on the voltage applied across it. We pick up one section of the LC line located at x (the next section will be located at $x + \Delta x$, and so on). Kirchhoff's current theorem and voltage theorem yield

$$\partial_x I + \frac{1}{\Delta x}\partial_t Q = 0 \qquad \frac{L}{\Delta x}\partial_t I_1 + \partial_x V = 0 \qquad \partial^2_{xt} V + \frac{1}{\Delta x C_S}(I - I_1) = 0 \tag{1}$$

where the current through the nonlinear capacitor is given by $\partial_t Q(V)$ (Q is the charge density). Using the quantity Δx which we assume to be small, we introduce dimensionless quantities defined by

$$\tilde{Q} = \frac{Q}{\Delta x} \qquad \tilde{L} = \frac{L}{\Delta x} \qquad \tilde{C}_S = \Delta x C_S.$$

Then (1) becomes

$$\partial_x I + \partial_t \tilde{Q}(V) = 0 \qquad \partial_x V + \tilde{L}\partial_t I_1 = 0 \qquad \partial^2_{xt} V + \frac{1}{\tilde{C}_S}(I - I_1) = 0$$

and we can write the set of partial differential equations for the voltages and currents as

$$\partial_x I + \partial_t Q(V) = 0 \qquad \partial_x V + L\partial_t I_1 = 0 \qquad \partial^2_{xt} V + \frac{1}{C_S}(I - I_1) = 0 \tag{1.1}$$

where Q stands for $\tilde{Q}$, L stands for $\tilde{L}$ and C_S stands for $\tilde{C}_S$. From (1.1) we can eliminate the currents I and I_1 and write

$$C_S \partial^4_{x^2t^2} V + \frac{1}{L}\partial^2_x V - \partial^2_t Q(V) = 0. \tag{1.2}$$

The simplest choice is to expand $Q(V)$ in a Taylor series as

$$Q(V) \approx C_0 V - C_N V^2.$$

Therefore (1.2) can be written as

$$\frac{1}{L}\partial^2_x V - C_0 \partial^2_t V + C_S \partial^4_{x^2t^2} V + C_N \partial^2_t V^2 = 0. \tag{1.3}$$

The first two terms in equation (1.3) are identical to the linear wave equation. The third term accounts for the dispersion introduced by the capacitor C_S, and the last term is the nonlinear term. If we neglect the nonlinear term and seek the solution of equation (1.3) in the form $V = \exp[\mathrm{i}(kx - \omega t)]$, we derive the following dispersion relation:

$$C_0 \omega_A^2 - \frac{1}{L}k_A^2 + C_S k_A^2 \omega_A^2 = 0. \tag{1.4}$$

In the next section, we derive two coupled Ginzburg–Landau equations describing the interaction of two wavepackets centred at (k_A, ω_A) and (k_B, ω_B). By setting $k_B = 0$ and $\omega_B = 0$, we deduce from the obtained CGLE a single nonlinear Schrödinger equation. The properties and stability of the solutions are studied in terms of the equations' coefficients (line coefficients) and k_j and ω_j, where $j = A, B$.

It is remarkable for the underlying model that coefficients of the CG-L system are complex, and this makes the mathematical studies a bit more complicated. We note that a single complex Ginzburg–Landau equation has been studied before by many authors (see [13–16]). Systems of Ginzburg–Landau equations with complex coefficients were investigated by, e.g., Cross [7], van Hecke [17] and Coullet and Frish [18].

The structure of this paper is as follows. The derivation of the CGLS and NLSE is outlined in section 2. In section 3 we use the coherent structure approach to analyse the existence and stability of the modulated amplitude waves (MAWs) [16] for the obtained CGLS. Section 4 is dedicated to the analysis of the modulational instability [11] of the periodic solution of the CGLE. Finally in section 5, the main results are summarized.

2. Derivation of the Ginzburg–Landau system

Nonlinearity is found almost everywhere in nature. The problem arises when one tries to solve the nonlinear equation that has been derived to describe the phenomena.

Considering the nonlinear equation (1.3) a system of Ginzburg–Landau equations can be derived using perturbation methods. To accomplish this, we consider the interaction of any two wavepackets centred at (k_A, ω_A) and (k_B, ω_B). To resolve the weakly nonlinear equation, two different slow time scales $T_1 = \epsilon t$, $T_2 = \epsilon^2 t$ must be introduced in addition to the original time scale $T_0 = t$. Moreover we introduce the large scale $X_1 = \epsilon x$ in addition to the original space scale $X_0 = x$. Here $0 < \epsilon \ll 1$. We note that the number of independent time scales needed depends on the order to which the expansion is carried out. In our work we carry out third-order expansion, and hence we need T_0, T_1 and T_2 (see [19]).

Then we seek a third-order solution in the form

$$V(x,t) = \epsilon V_1 + \epsilon^2 V_2 + \epsilon^3 V_3 + O(\epsilon^4) \tag{2.1}$$

where $V_n = V_n(X_0, X_1, T_0, T_1, T_2)$. To analyse the propagation of any two wavepackets centred at (k_A, ω_A) and (k_B, ω_B), we take the lowest order term V_1 in the form

$$V_1 = A\exp[\mathrm{i}(k_A X_0 - \omega_A T_0)] + B\exp[\mathrm{i}(k_B X_0 - \omega_B T_0)] + \mathrm{c.c.} \tag{2.2}$$

Here, the complex modulation amplitudes are functions of slow and space coordinates, i.e. $A = A(X_1, T_1, T_2)$ and $B = B(X_1, T_1, T_2)$.

Inserting the perturbation expansions (2.1) into the nonlinear equation (1.2), we obtain a series of non-homogeneous equations in different orders of ϵ:

$$\left(\frac{1}{L}\frac{\partial^2}{\partial X_0^2} - C_0\frac{\partial^2}{\partial T_0^2} + C_S\frac{\partial^4}{\partial X_0^2\partial T_0^2}\right)V_1 = D_0 V_1 = 0 \tag{2.3}$$

$$D_0 V_2 = 2C_0\frac{\partial^2 V_1}{\partial T_0\partial T_1} - C_N\frac{\partial^2 V_1^2}{\partial T_0^2} - \frac{2}{L}\frac{\partial^2 V_1}{\partial X_0\partial X_1} - 2C_S\left(\frac{\partial^4 V_1}{\partial X_0^2\partial T_0\partial T_1} + \frac{\partial^4 V_1}{\partial T_0^2\partial X_0\partial X_1}\right) \tag{2.4}$$

$$\begin{aligned} D_0 V_3 = {} & C_0\left(2\frac{\partial^2 V_1}{\partial T_0\partial T_2} + \frac{\partial^2 V_1}{\partial T_1^2} + 2\frac{\partial^2 V_2}{\partial T_0\partial T_1}\right) - 2C_N\left(\frac{\partial^2 V_1 V_2}{\partial T_0^2} + \frac{\partial^2 V_1^2}{\partial T_0\partial T_1}\right) \\ & - \frac{1}{L}\left(\frac{\partial^2 V_1}{\partial X_1^2} + 2\frac{\partial^2 V_2}{\partial X_0\partial X_1}\right) - C_S\left(2\frac{\partial^4 V_1}{\partial X_0^2\partial T_0\partial T_2} + \frac{\partial^4 V_1}{\partial X_0^2\partial T_1^2} + 4\frac{\partial^4 V_1}{\partial X_0\partial X_1\partial T_0\partial T_1}\right. \\ & \left. + 2\frac{\partial^4 V_1}{\partial X_0\partial X_2\partial T_0^2} + \frac{\partial^4 V_1}{\partial X_1^2\partial T_0^2} + 2\frac{\partial^4 V_2}{\partial X_0^2\partial T_0\partial T_1} + 2\frac{\partial^4 V_2}{\partial X_0\partial X_1\partial T_0^2}\right). \end{aligned} \tag{2.5}$$

Substituting (2.2) into (2.3) leads to the dispersion relations

$$\begin{cases} C_0\omega_A^2 - \frac{1}{L}k_A^2 + C_S k_A^2\omega_A^2 = 0 \\ C_0\omega_B^2 - \frac{1}{L}k_B^2 + C_S k_B^2\omega_B^2 = 0. \end{cases} \tag{2.6}$$

From equation (2.4) the following solvability conditions are obtained:

$$A_{T_1} = -v_{gA}A_{X_1} \qquad B_{T_1} = -v_{gB}B_{X_1} \tag{2.7}$$

where $v_{gA} = -\partial\omega_A/\partial k_A$ and $v_{gB} = -\partial\omega_B/\partial k_B$ are the group velocities. The subscripts T_1 and X_1 denote differentiation with respect to T_1 and X_1, respectively. It results from (2.7) that $A = A(X_A, T_2)$ and $B = B(X_B, T_2)$, where $X_A = X_1 - v_{gA}T_1$ and $X_B = X_1 - v_{gB}T_1$ are the shifted coordinates.

For the determination of V_2 we then obtain the following equation:

$$D_0 V_2 = -C_N \frac{\partial^2 V_1^2}{\partial T_0^2}$$

from where we have

$$V_2 = \frac{C_N}{3C_0 k_A^2} A^2\, \mathrm{e}^{2\mathrm{i}[k_A X_0 - \omega_A T_0]} + \frac{C_N}{3C_0 k_B^2} B^2\, \mathrm{e}^{2\mathrm{i}[k_B X_0 - \omega_B T_0]} + \frac{8C_N(\omega_A+\omega_B)^2}{(\omega_A+\omega_B)^2(C_0 + C_S(k_A+k_B)^2) - \frac{(k_A+k_B)^2}{L}} AB\, \mathrm{e}^{\mathrm{i}[(k_A+k_B)X_0 - (\omega_A+\omega_B)T_0]} + \frac{4C_N(\omega_A+\omega_B)^2}{(\omega_A+\omega_B)^2(C_0 + C_S(k_A+k_B)^2) - \frac{(k_A-k_B)^2}{L}} AB^*\, \mathrm{e}^{\mathrm{i}[(k_A-k_B)X_0 - (\omega_A-\omega_B)T_0]} + \text{c.c.} \tag{2.8}$$

where the asterisk stands for the complex conjugate.

If we insert (2.2) and (2.8) into (2.5) and eliminate the terms that produce secular terms in (2.5), we obtain the following system:

$$\mathrm{i}A_{T_2} - \frac{1}{2\omega_A} A_{T_1^2} + \frac{1}{C_0 + C_S k_A^2}\left[2C_S k_A A_{T_1 X_1} - \frac{C_S\omega_A - \frac{1}{L}}{2\omega_A} A_{X_1^2}\right] - \frac{C_N \omega_A}{C_0 + C_S k_A^2}(\tilde{a}|A|^2 + (\tilde{c}+\tilde{d})|B|^2)A = 0 \tag{2.9}$$

$$\mathrm{i}B_{T_2} - \frac{1}{2\omega_B} B_{T_1^2} + \frac{1}{C_0 + C_S k_B^2}\left[2C_S k_B B_{T_1 X_1} - \frac{C_S\omega_B - \frac{1}{L}}{2\omega_B} B_{X_1^2}\right] - \frac{C_N \omega_A}{C_0 + C_S k_B^2}(\tilde{b}|B|^2 + (\tilde{c}+\tilde{d})|A|^2)B = 0 \tag{2.10}$$

where

$$\tilde{a}_j = \frac{c_N}{3C_0 k_j^2} \qquad j = A, B$$
$$\tilde{c} = \frac{8C_N(\omega_A+\omega_B)^2}{(\omega_A+\omega_B)^2(C_0 + C_S(k_A+k_B)^2) - \frac{(k_A+k_B)^2}{L}} \tag{2.11}$$
$$\tilde{d} = \frac{4C_N(\omega_A-\omega_B)^2}{(\omega_A-\omega_B)^2(C_0 + C_S(k_A-k_B)^2) - \frac{(k_A-k_B)^2}{L}}.$$

It follows from (2.7) that

$$\frac{\partial^2 A}{\partial T_1^2} = -v_{gA}\frac{\partial^2 A}{\partial X_1 \partial T_1} = v_{gA}^2 \frac{\partial^2 A}{\partial T_1^2} \qquad \frac{\partial^2 A}{\partial T_1^2} = -v_{gB}\frac{\partial^2 B}{\partial X_1 \partial T_1} = v_{gB}^2 \frac{\partial^2 B}{\partial X_1^2}$$

which in (2.9) and (2.10) give

$$\mathrm{i}A_{T_2} - \left(\frac{v_{gA}^2}{2\omega_A} + \frac{2C_S k_A}{C_0 + C_S k_A^2} v_{gA} + \frac{C_S\omega_A - \frac{1}{L}}{2\omega_A\big(C_0 + C_S k_A^2\big)}\right) A_{X_1^2} - \frac{C_N\omega_A}{C_0 + C_S k_A^2}(\tilde{a}_A|A|^2 + (\tilde{c}+\tilde{d})|B|^2)A = 0$$

$$\mathrm{i}B_{T_2} - \left(\frac{v_{gB}^2}{2\omega_B} + \frac{2C_S k_B}{C_0 + C_S k_B^2} v_{gB} + \frac{C_S\omega_B - \frac{1}{L}}{2\omega_B\big(C_0 + C_S k_A^2\big)}\right) B_{X_1^2} - \frac{C_N\omega_A}{C_0 + C_S k_B^2}(\tilde{a}_B|B|^2 + (\tilde{c}+\tilde{d})|A|^2)B = 0.$$

Expressing X_1 and T_2 in terms of the original coordinates x and t, we obtain the following system:

$$\begin{cases} \partial_t A + \mathrm{i}\left(\frac{v_{gA}^2}{2\omega A} + \frac{2C_S k_A}{C_0 + C_S k_A^2} v_{gA} + \frac{C_S\omega_A - \frac{1}{L}}{2\omega_A\left(C_0 + C_S k_A^2\right)}\right)\partial_x^2 A + \mathrm{i}\frac{C_N\omega_A}{C_0 + C_S k_A^2}\epsilon^2(\tilde{a}_A|A|^2 + (\tilde{c} + \tilde{d})|B|^2)A = 0 \\ \partial_t B + \mathrm{i}\left(\frac{v_{gB}^2}{2\omega B} + \frac{2C_S k_B}{C_0 + C_S k_B^2} v_{gB} + \frac{C_S\omega_B - \frac{1}{L}}{2\omega_B\left(C_0 + C_S k_A^2\right)}\right)\partial_x^2 B + \mathrm{i}\frac{C_N\omega_B}{C_0 + C_S k_B^2}\epsilon^2(\tilde{a}_B|B|^2 + (\tilde{c} + \tilde{d})|A|^2)B = 0. \end{cases} \tag{2.12}$$

Using the dispersion relations (2.6), we put (2.12) in the form

$$\partial_t A - \frac{\mathrm{i}}{2}\frac{\mathrm{d}^2\omega_A}{\mathrm{d}k_A^2}\partial_x^2 A + \mathrm{i}\frac{C_N\omega_A}{C_0 + C_S k_A^2}\epsilon^2(\tilde{a}_A|A|^2 + (\tilde{c} + \tilde{d})|B|^2)A = 0 \tag{2.13$_A$}$$

$$\partial_t B - \frac{\mathrm{i}}{2}\frac{\mathrm{d}^2\omega_B}{\mathrm{d}k_B^2}\partial_x^2 B + \mathrm{i}\frac{C_N\omega_B}{C_0 + C_S k_B^2}\epsilon^2(\tilde{a}_B|B|^2 + (\tilde{c} + \tilde{d})|A|^2)B = 0. \tag{2.13$_B$}$$

These equations are known as coupled complex Ginzburg–Landau equations, referred to as the complex Ginzburg–Landau system (CGLS). It is important to realize that there are two $O(1)$ different group velocities, v_{gA} and v_{gB}, in this problem and therefore two frames of reference are used.

In what follows, we use the notation

$$\lambda_j = \frac{1}{2}\frac{\mathrm{d}^2\omega_j}{\mathrm{d}k_j^2} = -\frac{3}{2}\frac{LC_0C_S\omega_j^3}{C_0 + C_S k_j^2} \qquad Q_{jj} = \epsilon^2\frac{C_N\omega_j}{C_0 + C_S k_j^2}\widetilde{a}_j \qquad j = A, B$$
$$Q_{AB} = \epsilon^2\frac{C_N\omega_A}{C_0 + C_S k_A^2}(\tilde{c} + \tilde{d}) \qquad Q_{BA} = \epsilon^2\frac{C_N\omega_B}{C_0 + C_S k_B^2}(\tilde{c} + \tilde{d}) \tag{2.14}$$

and write system (2.13$_A$), (2.13$_B$) in the final form

$$\begin{cases} \partial_t A - \mathrm{i}\lambda_A\partial_x^2 A + \mathrm{i}(Q_{AA}|A|^2 + Q_{AB}|B|^2)A = 0 \\ \partial_t B - \mathrm{i}\lambda_B\partial_x^2 B + \mathrm{i}(Q_{BB}|B|^2 + Q_{BA}|A|^2)B = 0. \end{cases} \tag{2.15}$$

The diffusion coefficients λ_A and λ_B, and Q_A and Q_B are the coefficients of the Kerr nonlinearity. The diffusion coefficients λ_j measure the wave dispersion, and Q_j determine how the wave frequency is modulated. Here $j = A, B$.

By setting $B = 0$ ($k_B = \omega_B = 0$), we reduce system (2.15) to a single nonlinear Schrödinger equation

$$\partial_t A - \mathrm{i}P\partial_x^2 A + \mathrm{i}Q|A|^2 A = 0 \tag{2.16}$$

where $P = \lambda_A$ and $Q = Q_{AA}$. Next we also call equation (2.16) the complex Ginzburg–Landau equation. We note that $Q = Q_{AA} > 0$, $Q_{BB} > 0$, $\lambda_A < 0$, $\lambda_B < 0$ (if $\omega_j > 0$), and Q_{AB} and Q_{BA} have arbitrary sign.

The simplest nontrivial solutions to the CGLS (2.15) are the phase winding solutions (and also the plane wave solutions) of the form

$$\begin{cases} A(x,t) = a_0\exp\left[\mathrm{i}\left(q_A x - \left(\lambda_A q_A^2 + Q_{AA}a_0^2 + Q_{AB}b_0^2\right)t + \Omega_A^0\right)\right] \\ B(x,t) = b_0\exp\left[\mathrm{i}\left(q_B x - \left(\lambda_B q_B^2 + Q_{BB}a_0^2 + Q_{BA}b_0^2\right)t + \Omega_B^0\right)\right]. \end{cases} \tag{2.17}$$

Note that a phase solution to the CGLS (2.15) is a pair of functions (A, B) having the form

$$A(x,t) = a(t)\exp[\mathrm{i}(q_A x + \Omega_A(t))] \qquad B(x,t) = b(t)\exp[\mathrm{i}(q_B x + \Omega_B(t))]$$

for $(x,t) \in R \times R^+$ where a, b, Ω_A, Ω_B are real amplitudes and phases, respectively, depending on the time $t \in R^+$ only, and $q_A, q_B \in R$ are phase winding numbers. We note that (2.17) is the unique winding solution to CGLS (2.15). It follows from (2.17) that equation (2.16) has the following travelling wave solution

$$A(x,t) = a_0 \exp\left[\mathrm{i}\left(qx - \left(\lambda q^2 + Q a_0^2\right)t + \Omega_A^0\right)\right]. \tag{2.18}$$

If we assume that $k_A = 4\sqrt{\frac{189}{1613}}$ and $k_B = 6\sqrt{\frac{169}{9098}}$, then $\omega_A = 8\times 10^6$ and $\omega_B = 6\times 10^6$, and for line parameters

$$C_0 = 540\ \text{pF} \qquad C_S = 270\ \text{pF} \qquad C_N = 135\ \text{pF} \qquad L = 28\ \mu\text{H}$$

we compute the coefficients of system (2.15) with $\epsilon = 0.1$:

$$\begin{array}{lll} \lambda_A \approx -2996\,866.2528 & \lambda_B \approx -1835\,667.6375 & Q_{AA} \approx 458.8657 \\ Q_{BB} \approx 1327.1282 & Q_{AB} \approx 20\,575.8688 & Q_{BA} \approx 14\,937.265\,03. \end{array} \tag{2.19}$$

All the figures below correspond to data (2.19).

3. Existence and stability of MAWs

To analysis the existence and stability of MAWs for the CGLS (2.15), we first express A and B in the polar forms

$$A(x,t) = a(x,t)\exp[\mathrm{i}\varphi_A(x,t)] \qquad B(x,t) = b(x,t)\exp[\mathrm{i}\varphi_B(x,t)]. \tag{3.1}$$

Substituting (3.1) into (2.15) and separating real and imaginary parts, we have

$$\lambda_A\partial_x^2 a + \lambda_A a(\partial_x\varphi_A)^2 + a\partial_t\varphi_A + (Q_{AA}a^2 + Q_{AB}b^2)a = 0 \tag{3.2}$$

$$\lambda_B\partial_x^2 b + \lambda_B b(\partial_x\varphi_B)^2 + b\partial_t\varphi_B + (Q_{BB}b^2 + Q_{BA}a^2)b = 0 \tag{3.3}$$

$$\partial_t a + \lambda_A\left(2\partial_x a\partial_x\varphi_A + a\partial_x^2\varphi_A\right) = 0 \tag{3.4}$$

$$\partial_t b + \lambda_B\left(2\partial_x b \times \partial_x\varphi_B + b\partial_x^2\varphi_B\right) = 0. \tag{3.5}$$

The temporal evolution of coherent structures in the CGLS amounts to a uniform propagation with velocities v_A and v_B and an overall phase-oscillation with frequencies ω_A and ω_B:

$$a = a(z_A) \qquad b = b(z_B) \qquad \varphi_A(x,t) = \Phi_A(z_A) - \omega_A t \qquad \varphi_B(x,t) = \Phi_B(z_B) - \omega_B t \tag{3.6}$$

where $z_A = x - v_A t$, $z_B = x - v_B t$. Here $a(z_A)$, $\Phi_A(z_A)$, $b(z_B)$ and $\Phi_B(z_B)$ represent coherent structures. In what follows, we are interested only in the uniformly moving solutions.

Substitution of ansatz (3.6) into the system (3.2)–(3.5) yields the set of coupled nonlinear ordinary differential equations (ODEs):

$$-\lambda_A a'' + \lambda_A a(\Phi_A')^2 - v_A a\Phi_A' - \omega_A a + (Q_{AA}a^2 + Q_{AB}b^2)a = 0 \tag{3.7}$$

$$-\lambda_B b'' + \lambda_B b(\Phi_B')^2 - v_B b\Phi_B' - \omega_B b + (Q_{BB}b^2 + Q_{BA}a^2)b = 0 \tag{3.8}$$

$$-v_A a' + \lambda_A(2a'\Phi_A' + a\Phi_A'') = 0 \tag{3.9}$$

$$-v_B b' + \lambda_B(2b'\Phi_B' + b\Phi_B'') = 0. \tag{3.10}$$

Multiplying (3.9) by a and (3.10) by b and integrating the resulting equations yield

$$\lambda_A a^2\Phi_A' - \frac{v_A}{2}a^2 = c_A \qquad \lambda_B b^2\Phi_B' - \frac{v_B}{2}b^2 = c_B \tag{3.11}$$

respectively, where c_A and c_B are constants of integration. We shall assume in the following that $\Phi'_A = 0$ when $a = 0$ and $\Phi'_B = 0$ when $b = 0$ so that $c_A = c_B = 0$ and

$$\Phi'_j = \frac{v_j}{2\lambda_j} \qquad j = A, B \tag{3.12}$$

which when substituted into (3.7) and (3.8) give

$$a'' = -\frac{v_A^2 + 4\omega_A}{4\lambda_A} + \frac{Q_{AA}}{\lambda_A}a^3 + \frac{Q_{AB}}{\lambda_A}b^2 a \tag{3.13}$$

$$b'' = -\frac{v_B^2 + 4\omega_B}{4\lambda_B} + \frac{Q_{BB}}{\lambda_B}a^3 + \frac{Q_{BA}}{\lambda_B}a^2 b. \tag{3.14}$$

By setting $a' = X$ and $b' = Y$, we obtain the following four coupled nonlinear ODEs:

$$\begin{cases} a' = X \\ b' = Y \\ X' = -\frac{v_A^2 + 4\omega_A}{4\lambda_A}a + \frac{Q_{AA}}{\lambda_A}a^3 + \frac{Q_{AB}}{\lambda_A}b^2 a \\ Y' = -\frac{v_B^2 + 4\omega_B}{4\lambda_B}a + \frac{Q_{BB}}{\lambda_B}a^3 + \frac{Q_{BA}}{\lambda_B}a^2 b. \end{cases} \tag{3.15}$$

Next we examine stationary, i.e. time-independent, solutions to (3.15). Let us introduce the auxiliary function playing a crucial role in the stability analysis of (3.15):

$$G(v_A, v_B, \omega_A, \omega_B) = \frac{v_B^2 + 4\omega_B}{v_A^2 + 4\omega_A}. \tag{3.16}$$

Hereafter we will assume that the following conditions are satisfied:

$$\omega_j > 0 \qquad Q_{ij} > 0 \qquad D = Q_{AA}Q_{BB} - Q_{AB}Q_{BA} < 0 \qquad \text{for} \quad i, j = A, B. \tag{3.17}$$

Equation (3.17) implies that $G > 0$.

Let us denote the stationary amplitudes as

$$a_p = \sqrt{\frac{v_A^2 + 4\omega_A}{4Q_{AA}}} \qquad a_m = \sqrt{\frac{Q_{BB}\left(v_A^2 + 4\omega_A\right) - Q_{AB}\left(v_B^2 + 4\omega_B\right)}{4D}}$$

$$b_p = \sqrt{\frac{v_B^2 + 4\omega_B}{4Q_{BB}}} \qquad b_m = \sqrt{\frac{Q_{AA}\left(v_B^2 + 4\omega_B\right) - Q_{BA}\left(v_A^2 + 4\omega_A\right)}{4D}}.$$

Depending on the value of the function G generically three cases can occur:

1. $G < \frac{Q_{BB}}{Q_{AB}}$. In this case there are three non-negative stationary solutions: the zero solution (0, 0, 0, 0) and the pure modes (a_p, 0, 0, 0), (0, b_p, 0, 0). The solution (0, 0, 0, 0) and the pure mode (0, b_p, 0, 0) are unstable, and (a_p, 0, 0, 0) is an unstable fixed point.
2. $G > \frac{Q_{BA}}{Q_{AA}}$. In this case, there are three non-negative fixed points: the zero solution (0, 0, 0, 0) and the pure modes (a_p, 0, 0, 0), (0, b_p, 0, 0). The zero solution (0, 0, 0, 0) and the fixed point (a_p, 0, 0, 0) are unstable and (0, b_p, 0, 0) is a stable fixed point.
3. $\frac{Q_{BB}}{Q_{AB}} < G < \frac{Q_{BA}}{Q_{AA}}$. In this case there are four fixed points: the zero fixed point (0, 0, 0, 0), the pure modes (a_p, 0, 0, 0), (0, b_p, 0, 0) and the mixed mode (a_m, b_m, 0, 0). The pure modes (a_p, 0, 0, 0) and (0, b_p, 0, 0) are stable and the zero fixed point (0, 0, 0, 0) and the mixed point (a_m, b_m, 0, 0) are unstable.

Passing through the critical value $G_{c_1} = \frac{Q_{BB}}{Q_{AB}}$, the stable point ($a_p$, 0, 0, 0) becomes unstable and passing through the second critical value $G_{c_2} = \frac{Q_{BA}}{Q_{AA}}$ the stable fixed point (0, b_p, 0, 0, 0) becomes unstable.

4. Modulational instability

In this section we study the stability of the solution to the NLSE (2.16). If we express A in polar form $A(x,t) = \alpha(x,t)\exp[\mathrm{i}\varphi(x,t)]$ and substitute it in equation (2.16) and separate real and imaginary parts, we obtain

$$P\partial_x^2 a + Pa(\partial_x\varphi)^2 + a\partial_t\varphi + Qa^3 = 0 \tag{4.1}$$

$$\partial_t a + P\big(2\partial_x a\partial_x\varphi + a\partial_x^2\varphi\big) = 0. \tag{4.2}$$

System (4.1), (4.2) admits non-trivial solutions of the form

$$(a,\varphi) = \big(a_0, k_0x - \big(Pk_0^2 + Qa_0^2\big)t\big) \tag{4.3}$$

where $a_0 \neq 0$ and k_0 are real parameters. These solutions of the system (4.1), (4.2) correspond to the plane-wave solutions of the CGLE (2.16):

$$A(x,t) = a_x \exp\big[\mathrm{i}\big(k_0x - \big(Pk_0^2 + Qa_0^2\big)t\big)\big] \tag{4.4}$$

where a_0 and k_0 are amplitude and wavenumbers, respectively. Hereafter we call solutions (4.3) of the system (4.1), (4.2), and also solution (4.4) the plane-wave solution of the CGLE (2.16).

The linear stability of these solutions can be performed by considering the perturbed solution $\alpha(x,t) = (1+a_1)a_{0,}\varphi(x,t) = k_0x - \big(Pk_0^2 + Qa_0^2\big)t + v_1$, where $a_1 \propto a_{10}\exp[\mathrm{i}(Kx - \Omega(K)t)]$ and $\varphi_1 \propto \varphi_{10}\exp[\mathrm{i}(Kx - \Omega(K)t)]$. The growth rates associated with the complex perturbations a_1 and φ_1 is

$$\Omega(K) = 2Pk_0K \pm |P|K^2\left(1 + \frac{Q}{P}\frac{2a_0^2}{K^2}\right). \tag{4.5}$$

It follows from (4.5) that for a given $K \in R$, Ω is always real and a_1 and φ_1 are bounded if and only if $PQ > 0$. Otherwise Ω will be complex for values of $0 < |K| < K_c = |a_0|\sqrt{-2Q/P}$, and a_1 and φ_1 will be unbounded. Hence the plane-wave solutions of the CGLE (2.16) are always stable only if $PQ > 0$. For given P and Q, so that $PQ < 0$, the corresponding plane wave is linearly unstable against perturbations with wavenumbers K inside the interval $0 < |K| < K_c$.

However, linear stability is possible only for short times. The solution for long times is obtained in subsection 4.1 for the case of stable waves and in subsection 4.2 for the case of unstable waves.

4.1. Modulational stability

In this subsection, we consider the nonlinear modulation of the plane-wave solution (4.3) when $PQ > 0$. For this we introduce the new independent variable $z = x - vt$ and seek a solution for (4.1) and (4.2) in the form

$$a = a_0 + \alpha(z) \qquad \varphi(x,t) = \widetilde{\Phi}(z) + k_0x - \big(Pk_0^2 + Qa_0^2\big)t = \Phi(z) + \big(Pk_0^2 + Qa_0^2 - k_0v\big)t \tag{4.6}$$

where $z = x - vt$. If we introduce the notation $\Omega_0 = Qa_0^2 + Pk_0^2 - k_0v$ and insert (4.6) in (4.1), (4.2), we obtain

$$-P\alpha'' + Pa(\Phi')^2 - va\Phi' - \Omega_0 a + Qa^3 = 0 \tag{4.1*}$$

$$-va' + P(2\alpha'\Phi' + a\Phi'') = 0. \tag{4.2*}$$

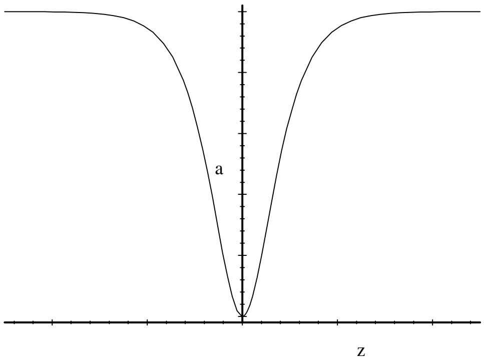

Figure 2. A soliton ($z = (x - \frac{v}{2P})l^{-1}$).

Multiplying (4.2*) by a and integrating the resulting equation yields

$$\Phi' a^2 = \frac{v}{2P} a^2 + c_1$$

where c_1 is a constant of integration. If we assume that $\Phi' = 0$ when $a = a_0$ we have $c_1 = -\frac{v}{2P} a_0^2$. Consequently,

$$\Phi'(z) = \frac{v}{2P}\left(1 - \frac{a_0^2}{a^2}\right)$$

which substituted into (4.1*) gives

$$a'' = \frac{v^2}{4P^2}\left(\frac{a_0^4}{a^3} - a\right) + \frac{Q}{P}\left(a^2 - a_0^2\right)a \tag{4.7}$$

if we take $k_0 = k_0(v) = v/P$.

A first integral of equation (4.7) is

$$(a')^2 = \frac{Q}{2P} a^2\left(a^2 - 2a_0^2\right) - \frac{v^2}{4P^2}\left(a^2 + \frac{a_0^4}{a^2}\right) + c_2$$

where c_2 is a constant of integration. In the special case where $a \to a_0$ and $a' \to 0$ as $z \to \infty$, we have $c_2 = \frac{Q}{4P} a_0^4 - \frac{v^2}{4P} a_0^2$. Therefore,

$$\frac{1}{2}(a')^2 = \frac{Q}{4P}\left(a^4 - a_0^4\right) - \frac{v^2}{8P^2}\left(a^2 - 2a_0^2\right) - \frac{v^2}{8P^2}\frac{a_0^4}{a^2} - \frac{Q}{2P} a_0^2 a^2. \tag{4.8}$$

Let $a^2 = \zeta$ and write (4.8) in the form

$$\frac{1}{4}(\zeta')^2 = \left(\frac{Q}{2P}\zeta - \left(\frac{v}{2P}\right)^2\right)(\zeta_0 - \zeta)^2. \tag{4.9}$$

For a soliton, $v^2 \leqslant 2PQ\zeta$. Hence $v \leqslant \sqrt{2PQ\zeta_{\min}}$, where $\zeta_{\min}$ is the minimum of a^2. Solving equation (4.9) we have the soliton solution, a wave consisting of a single dip of constant shape and speed, as shown in figure 2, of the form

$$a^2 = a_0^2 - \left(a_0^2 - \frac{v^2}{2PQ}\right)\frac{1}{\sec^2 \frac{z - \frac{v}{2P}t}{l}} \tag{4.10}$$

where

$$l = \sqrt{\frac{Q}{2P} a_0^2 - \left(\frac{v}{2P}\right)^2} \qquad v \leqslant a_{\min}\sqrt{\frac{Q}{2P}}. \tag{4.11}$$

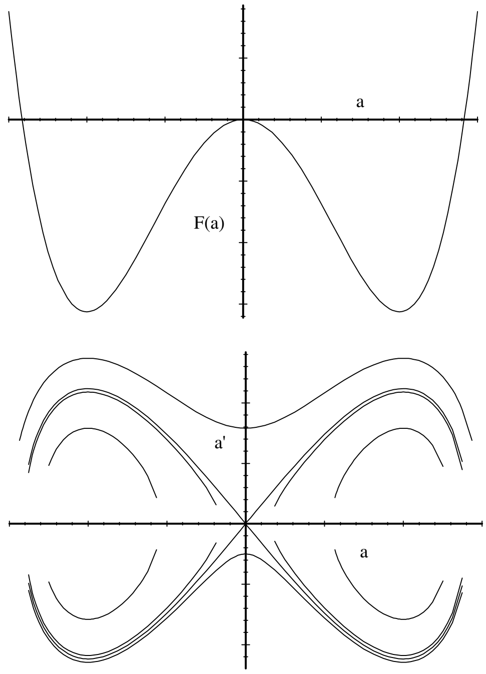

Figure 3. Phase plane for the case of modulational instability when v is zero.

4.2. Modulational instability

In this subsection the modulation of the envelope of the unstable plane-wave solution (4.3) is considered; that is, we consider solutions of (4.1*), (4.2*) when $PQ < 0$.

As in subsection 4.1, if we assume that $\Phi' = 0$ when $a = a_0$, we obtain that a is a solution of the ODE

$$a'' = \frac{Q}{P}(a^2 - a_0^2)a - \frac{v^2}{4P^2}\left(a - \frac{a_0^4}{a^3}\right). \tag{4.12}$$

Multiplying the last equation by a' and integrating the resulting equation gives

$$\frac{1}{2}(a')^2 = \frac{Q}{4P}(a^2 - 2a_0^2)a^2 - \frac{v^2}{8P^2}\left(a^2 + \frac{a_0^4}{a^2}\right) + h = -F(a) + h \tag{4.13}$$

where h is a constant of integration. From equation (4.12), we obtain the following system of ordinary differential equations:

$$\begin{cases} a' = b \\ b' = \frac{Q}{P}(a^2 - a_0^2)a - \frac{v^2}{4P^2}\left(a - \frac{a_0^4}{a^3}\right). \end{cases} \tag{4.14}$$

If $v = 0$, the system (4.14) admits three fixed points (0, 0), (a_0, 0) and ($-a_0$, 0). The solution (0, 0) is a saddle point, while ($-a_0$, 0) and (a_0, 0) are centres.

Figure 3 shows the character of the solutions in the phase plane for the case $v = 0$, with $h_0 = F(\pm a_0)$ and $h_1 = F(0)$. When $h = h_0$, there is no modulation of the wave motion. When $h_0 < h < h_1$, the modulation of the envelope is oscillatory about $a = \pm a_0$. The separatrix corresponds to $h = h_1$, which corresponds to a solution of soliton type (figure 4):

$$a = \frac{\sqrt{2}a_0}{\cosh\left[\sqrt{-\frac{2Q}{P}}a_0\left(x - \frac{v}{2P} - x_0 + \frac{v}{2P}t_0\right)\right]}$$

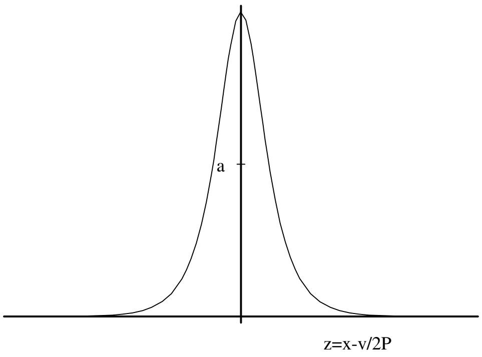

Figure 4. A soliton.

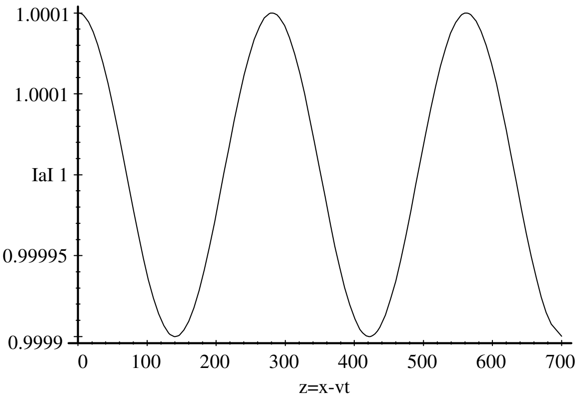

Figure 5. A profile of the stable stationary solution near $a = \pm a_0$.

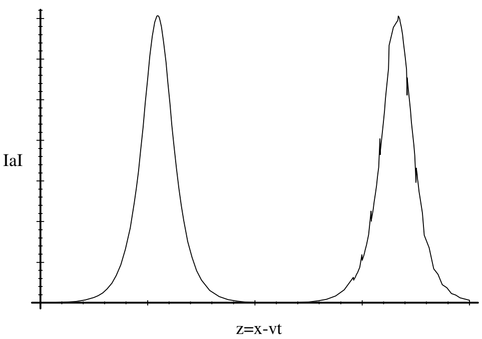

Figure 6. The profile of the unstable stationary solution featuring a cnoidal-wave-like solution.

where x_0 and t_0 are the initial values of x and t. When $h > h_1$, the modulation of the envelope is periodic, and a in this case passes through zero.

Figure 5 gives the profile of the stable stationary solution around the fixed points $a = \pm a_0$. As we can see from this figure, solution a oscillates around the fixed points $a = \pm a_0$. In figure 6, we present the profile of the unstable stationary solution, featuring a cnoidal-wave-like solution.

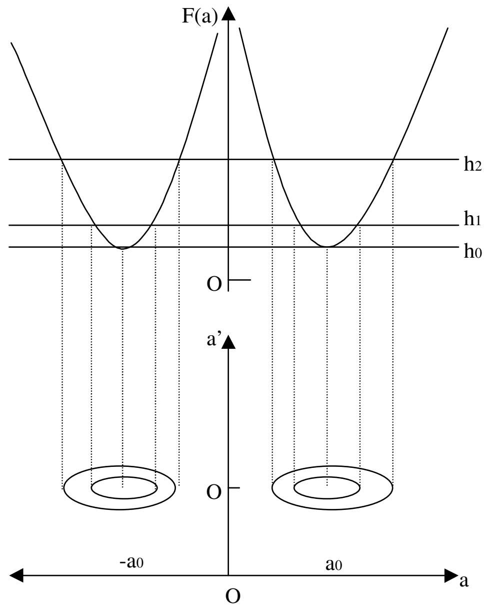

Figure 7. Phase plane for the case of modulational instability when $v \neq 0$.

Figure 7 shows the character of the solution in the phase plane for the case $v \neq 0$. In this case, there are two centres at $a = \pm a_0$ when $h = h_0 = F\,(\pm a_0)$, and there is no modulation. But when $h > h_0$, the modulation of the envelope is oscillatory about $a = \pm a_0$.

5. Conclusion

In the paper, we study the stability of a set of two coupled Ginzburg–Landau (GL) equations derived from a model of nonlinear dispersive-transmission line. The existence and stability of MAWs for the obtained CGLS are analysed and the existence of a soliton-like solution is proved. We have studied the stability or instability of the stationary wave solution for the obtained coupled Ginzburg–Landau system and have investigated the effect on the stability properties coming from the perturbation of both amplitude and phase. The modulational stability or instability of the plane-wave solution is also analysed.

A remarkable result of this paper is that the modulational instability or stability can be explained by the theory derived from the one-dimensional coupled Ginzburg–Landau system. Another surprising result of the paper is the existence of solitons on a nonlinear dispersive-transmission line: a soliton solution is shown to exist and be stable for a set of parameter values. This is the most interesting result for transmission lines. To our knowledge, the existence of such a stable solution is not known to engineers; let us hope that more will be known in the next few years.

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